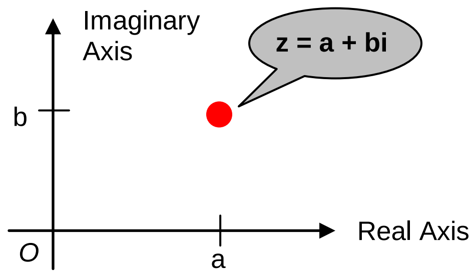
Section 6.5 (Complex Numbers in Polar Form: DeMoivre's Theorem)

A complex number z = a + bi can be represented as a point (a,b) in a coordinate plane, as shown in the figure



This coordinate system is called the complex plane and every complex number corresponds to a number in this plane

Example: Plot each number in the complex plane:

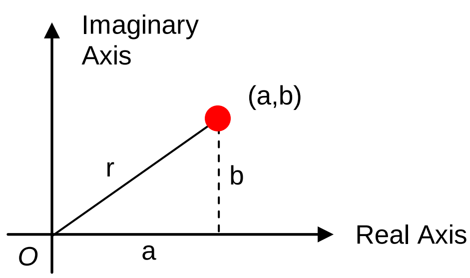
$z = 2 + 3i$ $z = -3 - 5i$ $z = -4$ $z = -i$

The absolute value of a complex number is the distance from the origin to that point in the complex plane (sketch line / triangle in figure to examine the distance from (a,b) to the origin)

$$|\mathbf{z}| = |\mathbf{a} + \mathbf{bi}| = \sqrt{\mathbf{a}^2 + \mathbf{b}^2}$$

Example: Determine the absolute value of each of z = 5 + 12i and 2 – 3i

A complex number of the form z = a + bi is said to be in rectangular form. Looking back at our figure, and letting r represent the distance (side) from point (a,b) to the origin, we have



$$\mathbf{r} = \sqrt{\mathbf{a}^2 + \mathbf{b}^2} \qquad \mathbf{cos}\ (\theta) = \frac{\mathbf{a}}{\mathbf{r}} \qquad \mathbf{sin}\ (\theta) = \frac{\mathbf{b}}{\mathbf{r}} \qquad \mathbf{tan}\ (\theta) = \frac{\mathbf{b}}{\mathbf{a}}$$

$$\mathbf{a = r\cos(\theta)} \qquad \mathbf{b = r\sin(\theta)}$$

Substituting back into rectangular form gives

$$z = a + bi = r\cos(\theta) + (r\sin(\theta))i = \mathbf{z = r\,(\cos(\theta) + i\sin(\theta))}$$

Example: Plot $\mathbf{z = -1 - i\sqrt{3}}$ in the complex plane and then express z in polar form (radians for angle measure)

To multiply 2 complex numbers in polar form, we multiply the moduli ($r_1 * r_2$) and add the arguments ($\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$) – see explanation and proof on pg. 690 in book

$$\mathbf{z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]}$$

Example: Find the product of the complex numbers given (leave answer in polar form)

$\mathbf{z_1 = 6(\cos(40^o) + i\sin(40^o))}$ $\mathbf{z_2 = 5(\cos(20^o) + i\sin(20^o))}$

Book problems:

To divide 2 complex numbers in polar form, we divide the moduli (r_1/r_2) and subtract the arguments ($\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)$) – see proof using algebraic complex number division in appendix in book if interested

Example: Find the quotient of the complex numbers given (leave answer in polar form)

$$z_1 = 50\left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right)\right) \qquad z_2 = 5\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)$$

We can also find powers of complex numbers in polar form using DeMoivre's Theorem. Consider z^2 …

$z = r(\cos(\theta) + i \sin(\theta))$ $\qquad$ $z^2 = r(\cos(\theta) + i \sin(\theta))\, r(\cos(\theta) + i \sin(\theta)) = r^2(\cos(2\theta) + i \sin(2\theta))$...

This pattern continues for higher powers of z in polar form meaning $\mathbf{z^n = r^n(\cos(n\theta) + i \sin(n\theta))}$

Example: Find $\mathbf{[2(\cos(30^o) + i \sin(30^o))]^5}$ and write the answer in rectangular form

Example: Find $\mathbf{(1 + i)^4}$ using DeMoivre's Theorem and write the answer in rectangular form

OPTIONAL ---

DeMoivre's Theorem can basically be reverse engineered to find complex roots as well (see in-depth discussion on pg. 693 in the book)...

Complex number $w = r(\cos(\theta) + i \sin(\theta))$ has n distinct nth roots that can be given as

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i \sin\left(\frac{\theta+2\pi k}{n}\right)\right] \qquad z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+360^o k}{n}\right) + i \sin\left(\frac{\theta+360^o k}{n}\right)\right]$$

where k = 0, 1, 2,…,n – 1 (note that each root has the same modulus)

Example: Find all the complex fourth roots of $\mathbf{16(\cos(60^o) + i \sin(60^o))}$ and write the roots in polar form

Example: Find all the cube roots of 27. Write roots in rectangular form…

Book problems: