Section 1.3 (Evaluating Limits Analytically)

In section 1.2, we introduced the concept of limits and learned that the limit of a function f(x) as x approaches c (a constant) isn't necessarily the function value at x = c, but oftentimes this is the case. If a function f(x) is **continuous at c**, then $\lim_{x\to c} f(x) = f(c)$. Here are some basic limits where b and c are real numbers and n is a positive integer…

$\lim_{x\to c} b = b$ $\qquad$ $\lim_{x\to c} x = c$ $\qquad$ $\lim_{x\to c} x^n = c^n$

(What do each of these look like graphically?)

Furthermore, with $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$ …

1. Scalar multiple: $\lim_{x\to c}[bf(x)] = bL$
2. Sum or difference: $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x\to c}[f(x)g(x)] = LK$
4. Quotient: $\lim_{x\to c}[f(x)/g(x)] = L/K$ (provided K ≠ 0)
5. Power: $\lim_{x\to c}[f(x)]^n = L^n$

Examples: Evaluate the following limits…

$\lim_{x\to 3}[-4]$ $\qquad$ $\lim_{x\to 2}[x]$ $\qquad$ $\lim_{x\to -3}[x]^3$ $\qquad$ $\lim_{x\to -2}[4x^3 - 3x]$

From the last example above, notice that the limit is the same as the function evaluated at x = – 2 . This simple substitution property is valid for all polynomials and rational functions with nonzero denominators. Consider polynomial functions p(x) and q(x) and rational function r(x) …

$\lim_{x\to c} p(x) = p(c)$ $\qquad$ $\lim_{x\to c} r(x) = r(c) = p(c)/q(c)$ (where q(c) ≠ 0)

Examples: Evaluate the following limits…

$\lim_{x\to 3}[3x^2 - 4]$ $\qquad$ $\lim_{x\to 2} \dfrac{x^2-3x+3}{x}$

Similar rules apply for roots and composite functions …

$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$ (valid for c>0 if n even) $\qquad$ $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)) = f(K)$

Homework problems: 5,8,13,18,27,32,37,43,45,51,55,65 (required) 48,61,85,103,104 (recommended)

Example: Evaluate the following limit…

$\lim_{x\to0}\sqrt{x^2+9}$ => g(x) = x^2 + 9, f(x) = $\sqrt{x}$ => $\lim_{x\to0}\sqrt{g(x)}$ => $\lim_{x\to0} f(g(x))$ => $f(\lim_{x\to0} x^2+9)$

Using direct substitution, we also have the trigonometric functions (assume c is in the domain of the function)…

$\lim_{x\to c} sin(x) = sin(c)$ $\qquad$ $\lim_{x\to c} cos(x) = cos(c)$ $\qquad$ $\lim_{x\to c} tan(x) = tan(c)$ …

Examples: Evaluate the following limits…

$\lim_{x\to0} sin(x)$ $\qquad$ $\lim_{x\to\pi} x\, cos(x)$ $\qquad$ $\lim_{x\to0} cos^2(x)$

Review factoring of **f(x) = x^3 – 1** and **g(x) = x^3 + 1** and discuss example 6 in the book…

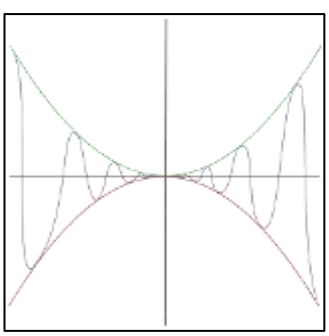
f(x) = x^3 – 1 = (x – 1)(x^2 + x + 1) $\qquad$ **g(x) = x^3 + 1 =**

Theorem 1.7: If a function f(x) = g(x) for all x ≠ c and the limit of g(x) as x -> c exists, then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$

Examples: Divide out factors and / or rationalize to find the following limits…

$\lim_{x\to-2} f(x) = \frac{x^2-3x-10}{x+2}$ $\qquad$ $\lim_{x\to0} g(x) = \frac{\sqrt{x+1}-1}{x}$

Review the squeeze theorem (sandwich, pinching) and examples in the book …



The limit $\lim_{x\to0} x^2 sin\left(\frac{1}{x}\right)$ cannot be determined through substitution / limit law because $\lim_{x\to0} sin\left(\frac{1}{x}\right)$ doesn't exist. However, we know that $-1 \le sin\left(\frac{1}{x}\right) \le 1$ and then $-x^2 \le x^2 sin\left(\frac{1}{x}\right) \le x^2$. Since $0 = \lim_{x\to0} x^2 = \lim_{x\to0}(-x^2)$, the squeeze theorem gives $\lim_{x\to0} x^2 = \lim_{x\to0} x^2 sin\left(\frac{1}{x}\right) = \lim_{x\to0}(-x^2)$, and $0 = \lim_{x\to0} x^2 sin\left(\frac{1}{x}\right) = 0$

Special Trig. Limits: $\lim_{x\to0}\left(\frac{sin(x)}{x}\right) = 1$ $\qquad$ $\lim_{x\to0}\frac{1-cos(x)}{x} = 0$

Homework problems: 5,8,13,18,27,32,37,43,45,51,55,65 (required) 48,61,85,103,104 (recommended)